Dual Quaternions

Dual Numbers

$$s = a + \varepsilon b \qquad where\ \varepsilon \neq 0, but\ \varepsilon^2 = 0 \tag{1}$$

$$s^{\dagger} = a - \varepsilon b \tag{2}$$

$$s s^{\dagger} = (a + \varepsilon b)(a - \varepsilon b) = a^2 \tag{3}$$

Dual Quaternion

$$\boldsymbol{Q} = \boldsymbol{r} + \varepsilon \boldsymbol{d} \tag{4}$$

Conjugate

There are multiple definitions for the conjugate of a dual quaternion.

$$\begin{aligned} \boldsymbol{Q}^{\dagger} &= \boldsymbol{r}^{\dagger} + \varepsilon \boldsymbol{d}^{\dagger} \\ \boldsymbol{Q}_{\varepsilon} &= r - \varepsilon \boldsymbol{d} \\ \boldsymbol{Q}_{\varepsilon}^{\dagger} &= r^{\dagger} - \varepsilon \boldsymbol{d}^{\dagger} \end{aligned} \tag{5}$$

The first definition is the one that will be useful to us. Recall that for regular (not dual) quaternions, we have

$$(\boldsymbol{p}\,\boldsymbol{q})^{\dagger} = \boldsymbol{q}^{\dagger}\,\boldsymbol{p}^{\dagger} \tag{6}$$

Then

$$\begin{aligned} \boldsymbol{Q}_1 \boldsymbol{Q}_2 &= (\boldsymbol{r}_1^{\dagger} + \varepsilon \boldsymbol{d}_1)(\boldsymbol{r}_2^{\dagger} + \varepsilon \boldsymbol{d}_2) \\ &= \boldsymbol{r}_1^{\dagger}\boldsymbol{r}_2^{\dagger} + \varepsilon \boldsymbol{r}_1^{\dagger}\boldsymbol{d}_2 + \varepsilon \boldsymbol{d}_1 \boldsymbol{r}_2^{\dagger} \\ &= \boldsymbol{r}_1^{\dagger}\boldsymbol{r}_2^{\dagger} + \varepsilon(\boldsymbol{r}_1^{\dagger}\boldsymbol{d}_2 + \boldsymbol{d}_1 \boldsymbol{r}_2^{\dagger}) \end{aligned}$$

$$\begin{aligned} \boldsymbol{Q}_2^{\dagger} \boldsymbol{Q}_1^{\dagger} &= (\boldsymbol{r}_2^{\dagger} + \varepsilon \boldsymbol{d}_2^{\dagger})(\boldsymbol{r}_1^{\dagger} + \varepsilon \boldsymbol{d}_1^{\dagger}) \\ &= \boldsymbol{r}_2^{\dagger}\boldsymbol{r}_1^{\dagger} + \varepsilon \boldsymbol{r}_2^{\dagger}\boldsymbol{d}_1^{\dagger} + \varepsilon \boldsymbol{d}_2^{\dagger}\boldsymbol{r}_1^{\dagger} \\ &= (\boldsymbol{r}_1 \boldsymbol{r}_2)^{\dagger} + \varepsilon(\boldsymbol{d}_1 \boldsymbol{r}_2)^{\dagger} + \varepsilon(\boldsymbol{r}_1 \boldsymbol{d}_2)^{\dagger} \\ &= (\boldsymbol{r}_1 \boldsymbol{r}_2)^{\dagger} + \varepsilon(\boldsymbol{d}_1 \boldsymbol{r}_2 + \boldsymbol{r}_1 \boldsymbol{d}_2)^{\dagger} \end{aligned} \tag{7}$$

So

$$(\boldsymbol{Q}_1 \boldsymbol{Q}_2)^{\dagger} = \boldsymbol{Q}_2^{\dagger} \boldsymbol{Q}_1^{\dagger} \tag{8}$$

Dual Quaternion for a Vector

$$\boldsymbol{V} = \boldsymbol{1} + \varepsilon \boldsymbol{v} \tag{9}$$

$$\boldsymbol{V}^{\dagger} = \boldsymbol{v}^{\dagger} = -\boldsymbol{V} \tag{10}$$

Dual Quaternion for Rotation then Translation

$$Q = r+\varepsilon\frac{1}{2}t\,r \qquad Q_\varepsilon^\dagger = r^\dagger - \varepsilon\frac{1}{2}(t\,r)^\dagger = r^\dagger - \varepsilon\frac{1}{2}r^\dagger t^\dagger \tag{11}$$

where $\mathbf{q_r}$ is the familiar rotation quaternion, and **t** is the translation vector. Applying this transformation to a vector gives:

$$\begin{aligned} Q\,V\,Q^\dagger &= (r+\varepsilon\frac{1}{2}t\,r)(\mathbf{1}+\varepsilon\,v)(r^\dagger-\varepsilon\frac{1}{2}r^\dagger t^\dagger) \\ &= (r+\varepsilon\,r\,v+\varepsilon\frac{1}{2}t\,r)(r^\dagger-\varepsilon\frac{1}{2}r^\dagger t^\dagger) \\ &= r r^\dagger - \varepsilon\frac{1}{2}r\,r^\dagger t^\dagger + \varepsilon\,r\,v\,r^\dagger + \varepsilon\frac{1}{2}t\,r\,r^\dagger \\ &= \mathbf{1} - \varepsilon\frac{1}{2}t^\dagger + \varepsilon\,r\,v\,r^\dagger + \varepsilon\frac{1}{2}t \end{aligned} \tag{12}$$

But **t** is a vector, so

$$-t^\dagger = +t \tag{13}$$

$$Q\,V\,Q^\dagger = \mathbf{1} + \varepsilon(r\,v\,r^\dagger + t) \tag{14}$$

which has the form of a rotation followed by a translation.

Dual Quaternion for Translation then Rotation

$$Q = r+\varepsilon\frac{1}{2}r\,t \qquad Q_\varepsilon^\dagger = r^\dagger - \varepsilon\frac{1}{2}(r\,t)^\dagger = r^\dagger - \varepsilon\frac{1}{2}t^\dagger r^\dagger \tag{15}$$

where $\mathbf{q_r}$ is the familiar rotation quaternion, and **d** is the translation vector. Applying this transformation to a vector gives:

$$\begin{aligned} Q\,V\,Q^\dagger &= (r+\varepsilon\frac{1}{2}r\,t)(\mathbf{1}+\varepsilon\,v)(r^\dagger-\varepsilon\frac{1}{2}t^\dagger r^\dagger) \\ &= (r+\varepsilon\,r\,v+\varepsilon\frac{1}{2}r\,t)(r^\dagger-\varepsilon\frac{1}{2}t^\dagger r^\dagger) \\ &= r r^\dagger - \varepsilon\frac{1}{2}r\,t^\dagger r^\dagger + \varepsilon\,r\,v\,r^\dagger + \varepsilon\frac{1}{2}r\,t\,r^\dagger \\ &= \mathbf{1} - \varepsilon\frac{1}{2}r\,t^\dagger r^\dagger + \varepsilon\,r\,v\,r^\dagger + \varepsilon\frac{1}{2}r\,t\,r^\dagger \\ &= \mathbf{1} + \varepsilon\,r(\frac{1}{2}t\ - \frac{1}{2}t^\dagger + v)r^\dagger \end{aligned} \tag{16}$$

But **t** is a vector, so

$$-t^\dagger = +t \tag{17}$$

$$Q\,V\,Q^\dagger = \mathbf{1} + \varepsilon\,r(v+t)r^\dagger \tag{18}$$

which has the form of a translation followed by a rotation. Finally, if d is the dual part of the quaternion,

$$d = \frac{1}{2} r\, t$$

$$r^{\dagger} d = \frac{1}{2} r^{\dagger} r\, t \tag{19}$$

$$r^{\dagger} d = \frac{\ }{2} t$$

So

$$t = 2 r^{\dagger} d \tag{20}$$

Using Dual Quaternions in Place of Vectors

In my application, most of the positions (vectors) I deal with are positions of objects. Since every object has a co-ordinate system associated with it (which may have been rotated), I find it easier to use only dual quaternions, instead of mixing dual quaternions and vectors.

Consider the case where two transforms in a row are applied to a vector.

$$Q_2 Q_1 V Q_1^{\dagger} Q_2^{\dagger} = Q_2 Q_1 V (Q_2 Q_1)^{\dagger} \tag{21}$$

So applying the transform Q_1 followed by the transform Q_2 is equivalent to applying the combined transform Q_1 Q_2. Or to look at it another way, we can consider Q_2 as transforming Q_1. So if we represent an object's position and rotation by Q_1, then if we apply the transform Q_2 to that, the result is:

$$Q_1 \rightarrow Q_2 Q_1 \tag{22}$$

Which means we have half as many operations to perform!